中国科学技术大学

复变函数A——习题解析

Chapter2

霍斌 PB24111627
计算机科学与技术学院
交流群:484030731

2025.9.22

请注意，本解题步骤主要依赖豆包对我手写内容进行识别，而后再经过人工查验生成，由于AI识别或许存在偏差以及本人解题时可能产生的错误，请谨慎参考。

1 Chapter2 - 复变数函数

1.1 1

$$\text{解}:(1)z=1+iy$$
$$\text{则}w=\frac{1}{1+iy}=\frac{1-iy}{1+y^2}=\frac{1}{1+y^2}-\frac{y}{1+y^2}i$$
是圆。

$$(2)z=x$$
$$w=\frac{1}{x}$$
是u轴，但不包括原点。

$$(3)z=x(1+i)$$
$$w=\frac{1}{z}=\frac{1}{x}\cdot\frac{1}{1+i}=\frac{1}{2x}(1-i)$$
是不包括原点的直线。

$$(4)z=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{4}-i\frac{y}{4}$$
是圆。

$$(5)x=1+\sqrt{5}\cos\theta,\quad y=\sqrt{5}\sin\theta$$
$$\implies z=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{4+2x}-i\frac{y}{4+2x}$$
由
$$\left(\frac{x}{4+2x}+\frac{1}{4}\right)^2+\left(\frac{y}{4+2x}\right)^2=\left(\frac{\sqrt{5}}{4}\right)^2$$

$$\text{或：}x^2+y^2-2x=4\implies |z|^2-2x=4$$
$$\implies \frac{1}{|w|^2}-\frac{2u}{u^2+v^2}=4\implies \frac{1-2u}{u^2+v^2}=4$$
$$\implies 4u^2+4v^2+2u-1=0$$
结论相同，知是以$\left(-\frac{1}{4},0\right)$为圆心，$\frac{\sqrt{5}}{4}$为半径的圆。

1.2 2

$$\text{解}:f(z)=\begin{cases}\frac{xy}{x^2+y^2}, & z\neq 0\\ 0, & z=0\end{cases}$$
$$\lim_{z\to 0}f(z)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}$$
令$y=kx$，则
$$\lim_{z\to 0}f(z)=\frac{kx^2}{x^2+k^2x^2}=\frac{k}{1+k^2}$$
当k发生变化，即$f(z)$以不同方向靠近原点，极限值不同，故不连续在$z=0$处。

1.3 3

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

$$= z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right)$$

当$z \to \infty$，括号内各项均$\to 0$（除a_n外），故$p_n(z) \to a_n z^n$。令$z = re^{i\varphi}$，故$|z^n| = r^n$。

当$z \to \infty$，$r \to \infty$，故$z^n \to \infty$。

a_n是个常数，故$a_n z^n \to \infty$。综上，$z \to \infty$时$p_n(z) \to \infty$。

1.4 4

考察$C-R$方程(1)令$f(z) = u(x,y) + iv(x,y)$，若可导即满足：

①当$z \neq 0$ $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ $f(z) = |z| = \sqrt{x^2+y^2}$，故$v(x,y) = 0$，即$v_x = v_y = 0$。而$u_x, u_y \neq 0$，故不可导。

②当$z = 0$时$\dfrac{\partial u}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$，分母为0，也不可导。综上$f(z) = |z|$不可导。

(2)类(1)，$u(x,y) = x + y$，$v(x,y) = 0$，不满足$C-R$方程，处处不可导。

(3)类(1)，$f(z) = \dfrac{1}{\bar{z}} = \dfrac{1}{x - iy} = \dfrac{x+iy}{|z|^2} = \dfrac{x}{x^2+y^2} + i\dfrac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{|z|^4}, \quad \frac{\partial v}{\partial y} = \frac{x^2 - y^2}{|z|^4}$$

不满足$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$，故$f(z) = \dfrac{1}{\bar{z}}$处处不可导。

1.5 5

$$(1)\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 1$$

由$C-R$方程

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \implies \begin{cases} y = 1 \\ x = -0 \end{cases} \implies \begin{cases} y = 1 \\ x = 0 \end{cases}$$

只在$z = i$这一个点可导，那么处处不解析。

$$(2)f(z) = \begin{cases} x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}, & |z| < 1 \\ x^2 - y^2 + i2xy, & |z| \geq 1 \end{cases}$$

①$|z| < 1$时，$u_x = \dfrac{2x^2+y^2}{|z|^2}$，$u_y = \dfrac{xy}{|z|^2}$

$$v_x = \frac{xy}{|z|^2}, \quad v_y = \frac{x^2+2y^2}{|z|^2}$$

$$\begin{cases} \frac{2x^2+y^2}{|z|^2} = \frac{x^2+2y^2}{|z|^2} \\ \frac{xy}{|z|^2} = -\frac{xy}{|z|^2} \end{cases} \implies x = y = 0 \implies z = 0$$

只有一点可导，故$|z| < 1$不解析。

②$|z| > 1$时，

$$\begin{cases} u_x = 2x = v_y \\ u_y = -2y = -v_x \end{cases}$$

处处解析。

③$|z| = 1$时，考察是否连续，若连续则

$$\lim_{|z| \to 1^-} f(z) = \lim_{|z| \to 1^-} z = \lim_{|z| \to 1^+} f(z) = \lim_{|z| \to 1^+} z^2 = f(z)\big|_{|z|=1}$$

只有$z = 1$时满足条件，所以$z = 1$处不解析。综上，解析区域是$|z| > 1$。

1.6 6

计算即可, 下面以第(2)题为例, 注意最后一步转化为$(1+z)e^z$

$$\text{解}(2): \text{设} f(z)=u+iv\text{，其中} u=e^x(x\cos y-y\sin y)\text{，} v=e^x(y\cos y+x\sin y)$$
$$u_x=e^x(x\cos y-y\sin y+\cos y)$$
$$u_y=e^x(-x\sin y-\sin y-y\cos y)$$
$$v_x=e^x(y\cos y+x\sin y+\sin y)$$
$$v_y=e^x(\cos y-y\sin y+x\cos y)$$
$$\begin{cases}u_x=v_y=e^x(x\cos y-y\sin y+\cos y)\\ u_y=-v_x=-e^x(y\cos y+x\sin y+\sin y)\end{cases}\text{且} u_x\text{，} u_y\text{，} v_x\text{，} v_y \text{在全平面上连续,}$$
所以$f(z)$在全平面上解析。
$$f'(z)=e^x(x\cos y-y\sin y+\cos y)+ie^x(y\cos y+x\sin y+\sin y)$$
$$=e^x[(x\cos y-y\sin y+\cos y)+i(y\cos y+x\sin y+\sin y)]$$
$$=e^x(1+x+iy)(\cos y+i\sin y)=(1+z)e^x(\cos y+i\sin y)=(1+z)e^z$$

1.7 7

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0},\quad g'(z_0)=\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}$$

$$\frac{f'(z_0)}{g'(z_0)}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)}=\lim_{z\to z_0}\frac{f(z)}{g(z)}\text{, 证毕。}$$

1.8 8

$$\text{解}: (1) f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0$$
又由$C-R$方程，可得$u_x=u_y=v_x=v_y=0$。故$f(z)=C_1+iC_2$

$(2)\overline{f(z)}=u(x,y)-iv(x,y)$类$(1)$，可证。

$(3)\mathrm{Re} f(z)=u(x,y)=C$，则由$C-R$方程类$(1)$可证。

(4)类(1)

$$(5)|f(z)|=\sqrt{u^2+v^2}=C$$
即$u^2(x,y)+v^2(x,y)=C^2$。则
$$\begin{cases}2uu_x+2vv_x=0\\ 2uu_y+2vv_y=0\end{cases}\implies\begin{cases}uu_x+vv_x=0\\ uu_y+vv_y=0\end{cases}$$
又由$C-R$方程$u_x=v_y$，$u_y=-v_x$，得
$$\begin{cases}uu_x+vv_x=0\\ vu_x-uv_x=0\end{cases}$$
视为一个齐次线性方程组，有
$$\begin{vmatrix}u & v\\ v & -u\end{vmatrix}=-(u^2+v^2)$$
要使此方程组有非平凡解，$u^2+v^2=|f(z)|^2=0$，也即当$|f(z)|\neq 0$时，$u_x=v_x=0$，$f(z)$为常数；当$|f(z)|=0$时，显然$f(z)=0$，为常数。

$$(6) f(z)=u(x,y)+iv(x,y)=\sqrt{u^2+v^2}\left(\frac{u}{\sqrt{u^2+v^2}}+i\frac{v}{\sqrt{u^2+v^2}}\right)$$
$$\arg f(z)=\arcsin\frac{v}{\sqrt{u^2+v^2}}=C\text{，故}$$
$$\frac{\partial\frac{v}{\sqrt{u^2+v^2}}}{\partial u}=\frac{u^2}{(u^2+v^2)^{\frac{3}{2}}}=0,\quad\frac{\partial\frac{v}{\sqrt{u^2+v^2}}}{\partial u}=\frac{-uv}{(u^2+v^2)^{\frac{3}{2}}}=0$$

$$\implies\begin{cases}u^2=0\\ -uv=0\end{cases}\implies u=0\implies v=0\Rightarrow f(z)=0$$

1.9 9

解：$\xi = f(z) = \xi(x,y) + i\eta(x,y)$

由$f(z)$是解析函数$\dfrac{\partial \xi}{\partial x} = \dfrac{\partial \eta}{\partial y},\ \dfrac{\partial \xi}{\partial y} = -\dfrac{\partial \eta}{\partial x}$

$$|f'(z)|^2 = \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 = \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2$$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial x},\ \frac{\partial H}{\partial y} = \frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial H}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial H}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$$= \frac{\partial^2 H}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 H}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial H}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}$$

$$+ \frac{\partial^2 H}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial^2 H}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial H}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$$\frac{\partial^2 H}{\partial y^2} = \frac{\partial^2 H}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 H}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial H}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}$$

$$+ \frac{\partial^2 H}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial^2 H}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial H}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}$$

$$LHS = \frac{\partial^2 H}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 H}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial^2 H}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 H}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2$$

$$+ 2\frac{\partial^2 H}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + 2\frac{\partial^2 H}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y} +$$

$$\frac{\partial H}{\partial \xi}\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + \frac{\partial H}{\partial \eta}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)$$

第一行 $= RHS$. 由$C-R$方程, 第二、三行 $= 0$,

故 $LHS = RHS$.

1.10 10

解：x-y中, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

令 $x = r\cos\theta,\ y = r\sin\theta$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x},\ \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y},\ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$dx = \cos\theta dr - r\sin\theta d\theta \implies dr = \cos\theta dx + \sin\theta dy$$

$$dy = \sin\theta dr + r\cos\theta d\theta \implies d\theta = \frac{-\sin\theta dx + \cos\theta dy}{r}$$

$$\implies \begin{cases} \frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \\ \frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \end{cases} \text{又} \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

可得 $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta},\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r}$

令 $z = re^{i\theta}$, 则 $f(z) = z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$

$u(r,\theta) = r^n\cos n\theta,\ v(r,\theta) = r^n\sin n\theta$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta,\ \frac{\partial u}{\partial \theta} = nr^{n-1}\sin n\theta,\ \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta,\ \frac{\partial v}{\partial \theta} = nr^n\cos n\theta$$

满足条件

对 $f(z) = \ln z = \ln r + i\theta$, 有

$u(r,\theta) = \ln r,\ v(r,\theta) = \theta$

$$\frac{\partial u}{\partial r} = \frac{1}{r},\ \frac{\partial u}{\partial \theta} = 0,\ \frac{\partial v}{\partial r} = 0,\ \frac{\partial v}{\partial \theta} = 1$$

满足条件

1.11 11

解：(1)

$\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}$, 只要 $z\neq 1$ 且 $z\neq 2$ 即可.

(2)

$\frac{1}{z^3+a}$, 只要 $z^3+a\neq 0$. 令 $z=re^{i\theta}$, 即 $r^3e^{i3\theta}=r^3(\cos 3\theta+i\sin 3\theta)=-a$

$$\Rightarrow \begin{cases} r^3=a \\ \cos 3\theta=-1 \\ \sin 3\theta=0 \end{cases} \Rightarrow \begin{cases} r=\sqrt[3]{a} \\ \theta=\frac{\pi+2k\pi}{3}, k=0,1,2 \end{cases}$$

$$\Rightarrow z\neq \sqrt[3]{a}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right), -\sqrt[3]{a}, \sqrt[3]{a}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

1.12 12

解：$\forall z_1\neq z_2\in D$ 有 $f(z_1)\neq f(z_2)$

不妨令 $f(z_1)=z_1^2+2z_1+3=z_2^2+2z_2+3=f(z_2)$, 则

$(z_1+z_2)(z_1-z_2)+2(z_1-z_2)=(z_1+z_2+2)(z_1-z_2)=0$

由 $z_1\neq z_2$ 知 $z_1+z_2+2=0$, 即 $z_1=-z_2-2$

显然若 $|z_2|<1$, 则 $|-z_2|<1$, 则

$|-z_2-2|>2-|z_2|\geq 1$, 与 $|z_1|<1$ 矛盾

故 $f(z)$ 为单叶映射

1.13 13

解: 上半虚轴为界，则有 $-\frac{3\pi}{2}<\arg z<\frac{\pi}{2}$.

令 $z=re^{i\theta}$，则当 $z=1$ 时，即

$w=\sqrt{r}e^{\frac{i(\theta+2k\pi)}{2}}=\cos\frac{\theta+2k\pi}{2}+i\sin\frac{\theta+2k\pi}{2}=\cos k\pi+i\sin k\pi$

由其在正实轴上取正值，知 $k=0$，即 $w=\sqrt{r}e^{\frac{i\theta}{2}}$

对左沿的 i，$\arg z\to -\frac{3\pi}{2}$，$w=e^{i(-\frac{3\pi}{4})}=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$

对右沿的 i，$\arg z\to \frac{\pi}{2}$，$w=e^{i(\frac{\pi}{4})}=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$

$w(-1)=e^{i(-\frac{\pi}{2})}=-i$, $w'(z)=\frac{1}{2\sqrt{z}}$，故 $w'(-1)=\frac{1}{-2i}=\frac{i}{2}$

1.14 14

解: 被开方项为零的点是函数的支点，只要令 $(1-z^2)(1-k^2z^2)=0$，可得

$z=\pm 1$ 或 $z=\pm\frac{1}{k}$，由 $0<k<1$，知 $\frac{1}{k}>1$，则支点顺序为：$-\frac{1}{k},-1,1,\frac{1}{k}$

正好是线段端点，因此能分出单值解析函数

由 $f(z)=\sqrt{|(1-z^2)(1-k^2z^2)|}\exp\left(i\frac{\arg((1-z^2)(1-k^2z^2))+ik\pi}{2}\right)$

当 $z=0$，$f(0)=\exp\left(i\frac{\arg(1)}{2}+ik\pi\right)>0$，则 $k=0$

故所求分支为 $f(z)=\sqrt{|(1-z^2)(1-k^2z^2)|}\exp\left(i\frac{\arg((1-z^2)(1-k^2z^2))}{2}\right)$

1.15 15

解：$w=f(z)=\sqrt[4]{z(1-z)^3}$，其支点满足 $z(1-z)^3=0$，则 $z=0$ 或 $z=1$.

正好在线段两端，故在其外有单值解析分支

对上沿，令 $z=x+i\varepsilon$，其中 $\varepsilon\to 0^+$，对 $z=x$，$\arg(z)=0$ 且 $\arg(1-z)=0$.

$\arg w=\frac{1}{4}(\arg(z)+3\arg(1-z))=0$，则 $w(x)=\sqrt[4]{x(1-x)^3}>0$.

$\implies \arg(w(-1))=\frac{1}{4}[\arg(-1)+3\arg(2)]=\frac{\pi}{4}$, $|w(-1)|=\sqrt[4]{8}$.

$\implies w(-1)=\sqrt[4]{8}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$.

$\ln w=\frac{1}{4}[\ln z+3\ln(1-z)]$ 故

$\frac{w'(z)}{w(z)}=\frac{1-4z}{4z(1-z)} \implies w'(-1)=\sqrt[4]{2}(1+i)\cdot\left(-\frac{5}{8}\right)=\frac{-5\sqrt[4]{2}}{8}(1+i)$

1.16 16

解：(1)令 $z=x+iy$，则 $\dfrac{z}{e^z}=\dfrac{x+iy}{e^xe^{iy}}=\dfrac{x+iy}{e^x(\cos y+i\sin y)}=\dfrac{x\cos y+y\sin y+i(y\cos y-x\sin y)}{e^x}$

当$z\to\infty$，不妨令 $x=0$, $y\to+\infty$，则

$\lim\limits_{z\to\infty}\dfrac{z}{e^z}=\lim\limits_{y\to+\infty}y(\sin y+i\cos y)$，故无极限

(2) $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$，则 $z\sin\dfrac{1}{z}=\dfrac{z}{2i}\left(e^{\frac{i}{z}}-e^{\frac{-i}{z}}\right)$，不妨令 $z=x+iy$

则 $z\sin\dfrac{1}{z}=\dfrac{x+iy}{2i}\left(e^{\frac{y+ix}{x^2+y^2}}-e^{\frac{-y-ix}{x^2+y^2}}\right)=\dfrac{y-ix}{2}\left[\cos\dfrac{x}{x^2+y^2}\left(e^{\frac{y}{x^2+y^2}}-e^{\frac{-y}{x^2+y^2}}\right)+i\left(e^{\frac{y}{x^2+y^2}}\sin\dfrac{x}{x^2+y^2}+e^{\frac{-y}{x^2+y^2}}\sin\dfrac{x}{x^2+y^2}\right)\right]$

$\implies \left|z\sin\dfrac{1}{z}\right|=\sqrt{\dfrac{x^2+y^2}{2}}\sqrt{e^{\frac{2y}{x^2+y^2}}+e^{\frac{-2y}{x^2+y^2}}-2\cos\dfrac{2x}{x^2+y^2}}$

不妨令 $x=y=\varepsilon\to0^+$，则 $\left|z\sin\dfrac{1}{z}\right|\to\dfrac{e}{\sqrt{2}}e^{\frac{1}{2\varepsilon}}=+\infty$，故不存在

(3)令 $z=1+re^{i\varphi}$, $r\to0^+$

分母$e^z\to e^{-1}$，分子$z\to1$，另一个因子：

$\dfrac{1}{z-1}=\dfrac{1}{r}e^{-i\varphi}=\dfrac{1}{r}(\cos\varphi-i\sin\varphi)$，则 $e^{\frac{1}{z-1}}=e^{\frac{\cos\varphi}{r}}\cdot e^{\frac{-\sin\varphi}{r}i}$

不妨令 $\varphi=0$，则 $e^{\frac{1}{z-1}}=e^{\frac{1}{r}}\to+\infty$，若令 $\varphi=\dfrac{\pi}{2}$，则 $e^{\frac{1}{z-1}}=e^{\frac{-i}{r}}$，不定的，故不存在

1.17 17

解：不妨令 $z=x+iy$，则 $y=kx$ (射线不垂直于x轴时)

$z+e^z=x+ikx+e^x(\cos kx+i\sin kx)$，若 $k\neq0$，则 $z+e^z\to\infty$

若 $k=0$，则 $z+e^z=x+e^x\to\infty$

若射线垂直于x轴，则 $z=Ai$, $A\to\infty$，则

$z+e^z=Ai+e^{Ai}=\cos A+i(\sin A+A)\to\infty$

1.18 18

解：$\sin z=2$，即$\dfrac{e^{iz}-e^{-iz}}{2i}=2$，即$e^{iz}-e^{-iz}=4i$，即

$e^{i2z}-1=4ie^{iz}$，即$e^{i2z}-4ie^{iz}-1=0$，令 $t=e^{iz}$，即

$t^2-4it-1=0$，则 $t=\dfrac{4i\pm\sqrt{-16+4}}{2}=(2\pm\sqrt{3})i=e^{iz}$

两边取对数，有 $iz=\ln(i(2\pm\sqrt{3}))=\ln(2\pm\sqrt{3})+i\arg(i(2\pm\sqrt{3}))+i2k\pi$

即$z=-i\ln(2\pm\sqrt{3})+\dfrac{\pi}{2}+2k\pi$

1.19 19

解：(1)令 $1+e^z=0$, $z=x+iy$，则 $e^xe^{iy}=e^x(\cos y+i\sin y)=-1$

则 $x=0$, $\cos y=-1$, $\sin y=0 \implies y=\pi+2k\pi$

$\implies D=\{x+iy\mid y\neq\pi+2k\pi,\ k\in\mathbb{Z}\}$

微商：$f'(z)=\dfrac{-e^z}{(1+e^z)^2}$

(2) $\sin z=2$，即$z=\dfrac{\pi}{2}+2k\pi-i\ln(2\pm\sqrt{3})$

$\implies D=\left\{z\mid z\neq\dfrac{\pi}{2}+2k\pi-i\ln(2\pm\sqrt{3})\right\}$ (见本章18)

微商：$f'(z)=\dfrac{-\cos z}{(\sin z-2)^2}$

(3)只要$z\neq1$ 即可，$D=\{z\mid z\neq1\}$

微商：$f'(z)=e^{\frac{1}{z-1}}-\dfrac{ze^{\frac{1}{z-1}}}{(z-1)^2}$

1.20 20

解: (1) $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$,

$LHS = \dfrac{e^{iz_1+iz_2}+e^{-iz_1-iz_2}}{2}$,

$RHS = \left(\dfrac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\dfrac{e^{iz_2}+e^{-iz_2}}{2}\right)-\left(\dfrac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\dfrac{e^{iz_2}-e^{-iz_2}}{2i}\right)$

$= \dfrac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}+e^{i(z_1-z_2)}+e^{-i(z_1-z_2)}+e^{z_1+z_2}-e^{z_1-z_2}+e^{-(z_1+z_2)}-e^{-(z_1-z_2)}}{4} = LHS$

(2)类(1)，只需硬算即可，此处略. 1.21

(3)令 $w = \text{Arccos}z$，则 $z = \cos w = \dfrac{e^{iw}+e^{-iw}}{2}$，即$e^{iw}+e^{-iw}-2z=0$，即

$e^{2iw}-2ze^{iw}+1=0$，令 $t=e^{iw}$，即$t^2-2zt+1=0$，有

$t = \dfrac{2z\pm\sqrt{4z^2-4}}{2} = z\pm\sqrt{z^2-1} = e^{iw}$，即

$iw = \ln\left(z\pm\sqrt{z^2-1}\right)$，即$w = -i\ln\left(z+\sqrt{z^2-1}\right)$

(此处$\sqrt{z^2-1}$是一个双值函数，替代了“$\pm$”的作用，详见课本).

1.21 21

解：$\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$，令 $z=x+iy$，则

$\sin z = \dfrac{e^{-y+ix}-e^{y-ix}}{2i} = \dfrac{e^{-y}(\cos x+i\sin x)-e^{y}(\cos x-i\sin x)}{2i}$

$= \dfrac{e^{-y}\sin x+e^{y}\sin x}{2}+i\left(\dfrac{-e^{-y}\cos x+e^{y}\cos x}{2}\right)$

$\Longrightarrow$ 实部：$\dfrac{\sin x(e^{y}+e^{-y})}{2} = \sin x\cosh y$

虚部：$\dfrac{\cos x(e^{y}-e^{-y})}{2} = \cos x\sinh y$

模：$\sqrt{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y}$

1.22 22

解: 类 21，可求出 $\cos z$ 的虚部为 $-\sinh y\sin x$

只要其为 0，即 $\left(\dfrac{e^{z}-e^{-z}}{2}\right)\sin x = 0$

① $x=0$,

② $x\neq 0$, $e^{z}-e^{-z}=0$，即$e^{x}(\cos y+i\sin y) = e^{-x}(\cos y-i\sin y)$

$e^{2x} = \cos 2y - i\sin 2y$，只能 $\sin 2y = 0$

$\Longrightarrow\ 2y = \pi+k\pi \ \Longrightarrow\ y = \dfrac{\pi}{2}+\dfrac{k\pi}{2}$

故在$x=0$ 与 $y=\dfrac{\pi}{2}+\dfrac{k\pi}{2}$, $k\in\mathbb{Z}$ 上取实数

1.23 23

解: 有 $\mathrm{Ln}z = \ln|z| + i(\arg z + 2k\pi)$，故

$\mathrm{Ln}(-1) = i(\pi + 2k\pi),\ k \in \mathbb{Z}$ （arg 表主幅角，范围 $(-\pi, \pi]$）

$\ln(-1) = \pi$

$\mathrm{Ln}(i) = i\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z}$

$\ln(i) = i\frac{\pi}{2}$

$\mathrm{Ln}(3-2i) = \ln\sqrt{13} + i\left(\arctan\frac{-2}{3} + 2k\pi\right)$

$\mathrm{Ln}(-2+3i) = \ln\sqrt{13} + i\left(\arctan\frac{3}{-2} + \pi\right)$

$1^{\sqrt{2}} = e^{\sqrt{2}\mathrm{Ln}1} = e^{\sqrt{2}(2k\pi i)} = e^{i2\sqrt{2}k\pi} = \cos(2\sqrt{2}k\pi) + i\sin(2\sqrt{2}k\pi)$

$(-2)^{\sqrt{2}} = e^{\sqrt{2}\mathrm{Ln}(-2)} = e^{\sqrt{2}(\ln 2 + i(\pi+2k\pi))} = e^{\sqrt{2}\ln 2}e^{i\sqrt{2}(\pi+2k\pi)} = e^{\sqrt{2}\ln 2}(\cos(\sqrt{2}\pi) + i\sin(\sqrt{2}\pi)) = -e^{\sqrt{2}\ln 2}$

$2^i = e^{i\mathrm{Ln}2} = e^{i(\ln 2 + 2k\pi i)} = e^{-2k\pi + i\ln 2} = e^{-2k\pi}(\cos(\ln 2) + i\sin(\ln 2))$

$\cos(2+i) = \dfrac{e^{-1+2i} + e^{1-2i}}{2} = \dfrac{(e^2+1)\cos 2}{2e} + i\dfrac{(1-e^2)\sin 2}{2e}$

$\sin 2i = \dfrac{e^{-2} - e^2}{2i} = \dfrac{i}{2}(e^2 - e^{-2})$

$\cot\left(\frac{\pi}{4} - i\ln 2\right) = \dfrac{\cos\left(\frac{\pi}{4} - i\ln 2\right)}{\sin\left(\frac{\pi}{4} - i\ln 2\right)} = \dfrac{\left(e^{\ln 2 + i\frac{\pi}{4}} + e^{-\ln 2 - i\frac{\pi}{4}}\right)2i}{e^{\ln 2 + i\frac{\pi}{4}} - e^{-\ln 2 - i\frac{\pi}{4}}} =$

$\dfrac{\left[e^{\ln 2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + e^{\ln\frac{1}{2}}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right]2i}{e^{\ln 2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) - e^{\ln\frac{1}{2}}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)} = \dfrac{(2i)(\sqrt{2}+i\sqrt{2})(\sqrt{2}-i\sqrt{2})}{4\left[\frac{1}{4}(\sqrt{2}+i\sqrt{2}) - \frac{1}{4}(\sqrt{2}-i\sqrt{2})\right]} = \dfrac{\sqrt{2}}{17}(5+3i)$

使用 $\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$

$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$

有 $\coth(2+i) = \dfrac{\cosh 2\cos 1 + i\sinh 2\sin 1}{\sinh 2\cos 1 + i\cosh 2\sin 1} = \dfrac{\sinh 4 - i\sin 2}{\cosh 4 - \cos 2}$

使用 $\mathrm{Arcsin}z = -i\ln(iz + \sqrt{1-z^2})$ 与 $\mathrm{Arccos}z = -i\ln(z + \sqrt{z^2-1})$ 即可得

(前者见书本，后者见本章习题 20.(3))

1.24 24

解：$(ab)^c$与 a^{bc}

直接举反例即可证不相等：取$a=-1,\ b=2,\ c=\dfrac{1}{2}$,

$(ab)^c = \sqrt{1} = \pm 1$, 而$a^{bc} = (-1)^1 = -1$

故 $(ab)^c$不一定等于 a^{bc}