中国科学技术大学

复变函数A——习题解析

Chapter1

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请注意，本解题步骤主要依赖豆包对我手写内容进行识别，而后再经过人工查验生成，由于AI识别或许存在偏差以及本人解题时可能产生的错误，请谨慎参考。

1 Chapter1 - 复数与平面点集

1.1

$$\text{解}:(1)z=2-2i=2\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=2\sqrt{2}\left(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})\right)=2\sqrt{2}e^{-i\frac{\pi}{4}}$$

$$\mathrm{Arg}z=-\frac{\pi}{4}+2k\pi,k\in\mathbb{Z}$$

$$(2)z=-\sqrt{3}i=\sqrt{3}(0-i)=\sqrt{3}\left(\cos(-\frac{\pi}{2})+i\sin(-\frac{\pi}{2})\right)=\sqrt{3}e^{-i\frac{\pi}{2}}$$

$$\mathrm{Arg}z=-\frac{\pi}{2}+2k\pi,k\in\mathbb{Z}$$

$$(3)z=\frac{\sqrt{13}}{2}\left(-\frac{1}{\sqrt{13}}-\frac{2\sqrt{3}}{\sqrt{13}}\right)=\frac{\sqrt{13}}{2}\left(\arccos(-\frac{1}{\sqrt{13}})+\arcsin(-\frac{2\sqrt{3}}{\sqrt{13}})\right)=\frac{\sqrt{13}}{2}e^{i(-\pi+\arctan 2\sqrt{3})}$$

$$\mathrm{Arg}z=(2k-1)\pi+\arctan 2\sqrt{3},k\in\mathbb{Z}$$

$$(4)z=1-\cos\theta+i\sin\theta=2\left|\sin\frac{\theta}{2}\right|e^{i(\frac{\pi}{2}-\frac{\theta}{2})}$$

$$\mathrm{Arg}z=\frac{\pi}{2}-\frac{\theta}{2}+2k\pi,k\in\mathbb{Z}$$

$$\text{解}:(1)z=2-2i=2\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=2\sqrt{2}\left(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})\right)=2\sqrt{2}e^{-i\frac{\pi}{4}}$$

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$$\mathrm{Arg}z=(2k-1)\pi+\arctan 2\sqrt{3},k\in\mathbb{Z}$$

$$(4)z=1-\cos\theta+i\sin\theta=2\left|\sin\frac{\theta}{2}\right|e^{i(\frac{\pi}{2}-\frac{\theta}{2})}$$

$$\mathrm{Arg}z=\frac{\pi}{2}-\frac{\theta}{2}+2k\pi,k\in\mathbb{Z}$$

1.2

$$\text{解：（1）令}z=re^{i\varphi}\text{，则}z^3=r^3(\cos 3\varphi+i\sin 3\varphi)$$

$$r^3=\sqrt{1+3}=2\Rightarrow r=\sqrt[3]{2}\text{，}$$

$$\begin{cases}\cos 3\varphi=-\frac{1}{2}\\ \sin 3\varphi=\frac{\sqrt{3}}{2}\end{cases}\Rightarrow 3\varphi=\frac{2\pi}{3}+2k\pi\Rightarrow\varphi=\frac{\frac{2\pi}{3}+2k\pi}{3}$$

$$\Rightarrow z=\sqrt[3]{2}\left[\cos\frac{\frac{2\pi}{3}+2k\pi}{3}+i\sin\frac{\frac{2\pi}{3}+2k\pi}{3}\right],k=0,1,2$$

（2）略

$$\text{（3）}z^4=r^4(\cos 4\varphi+i\sin 4\varphi)=-1$$

$$\begin{cases}\cos 4\varphi=-1\\ \sin 4\varphi=0\end{cases}\Rightarrow 4\varphi=\pi+2k\pi\Rightarrow\varphi=\frac{\pi}{4}+\frac{k\pi}{2}$$

$$\Rightarrow z=\cos\frac{\pi+2k\pi}{4}+i\sin\frac{\pi+2k\pi}{4},k=0,1,2,3$$

1.3

$$\text{由立方根公式知道} w^3-1=(w-1)(w^2+w+1)=0$$
$$\text{由} w \text{是复根，知} w-1\neq 0, \text{故} w^2+w+1=0 \text{解：由立方根公式知道} w^3-1=(w-1)(w^2+w+1)=0$$
$$\text{由} w \text{是复根，知} w-1\neq 0, \text{故} w^2+w+1=0$$

1.4

$$\text{解：对等式两边平方, 有}$$
$$x^2-y^2+2xyi=a+bi$$
$$\Rightarrow \begin{cases} a=x^2-y^2 \\ b=2xy \end{cases} \Rightarrow \qquad \begin{cases} x=\pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} \\ y=\pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}} \end{cases}$$
$$b>0,\ xy \text{ 同号；} b<0,\ xy \text{ 异号}$$

1.5

$$\text{令} S_n=\sum_{k=1}^{n} e^{ik\theta}=\sum_{k=1}^{n}(\cos k\theta+i\sin k\theta)$$
$$\text{等比数列首项} a=e^{i\theta}\text{，公比为} r=e^{i\theta}$$
$$S_n=a\frac{1-r^n}{1-r}=e^{i\theta}\frac{1-e^{in\theta}}{1-e^{i\theta}}$$
$$1-e^{in\theta}=e^{i\frac{n\theta}{2}}\left(e^{-i\frac{n\theta}{2}}-e^{i\frac{n\theta}{2}}\right)=e^{i\frac{n\theta}{2}}\left(-2i\sin\frac{n\theta}{2}\right)=-2ie^{i\frac{n\theta}{2}}\sin\frac{n\theta}{2}$$
$$1-e^{i\theta}=e^{i\frac{n\theta}{2}}\left(-2i\sin\frac{n\theta}{2}\right)\bigg|_{n=1}=-2ie^{i\frac{\theta}{2}}\sin\frac{\theta}{2}$$
$$\Rightarrow S_n=e^{i\theta}\frac{-2ie^{i\frac{n\theta}{2}}\sin\frac{n\theta}{2}}{-2ie^{i\frac{\theta}{2}}\sin\frac{\theta}{2}}=e^{i\left(\frac{n+1}{2}\right)\theta}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}=\left[\cos\left(\frac{n+1}{2}\right)\theta+i\sin\left(\frac{n+1}{2}\right)\theta\right]\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$
$$\Rightarrow \mathrm{Re}(S_n)=\frac{\cos\left(\frac{n+1}{2}\right)\theta\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}=-\frac{1}{2}+\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$
$$\text{即（1）证毕，对（2），} \mathrm{Im}(S_n)=\sin\left(\frac{n+1}{2}\right)\theta\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\text{，类（1）易证（2）}$$

1.6

$$\text{解：}|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\mathrm{Re}(z_1\overline{z_2})$$
$$|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\mathrm{Re}(z_1\overline{z_2})$$
$$\Rightarrow \mathrm{LHS}=2(|z_1|^2+|z_2|^2)=\mathrm{RHS}$$
$$\text{对于平行四边形的四个边a,b,c,d(按照逆时针顺序)有} 2(a^2+b^2)=c^2+d^2$$

1.7

$$\text{解}:(1)\ \left|\frac{z-a}{1-\overline{a}z}\right|=\frac{|z-a|}{|1-\overline{a}z|}$$
$$|z-a|^2=|z|^2+|a|^2-2\mathrm{Re}(z\overline{a})=1+|a|^2-2\mathrm{Re}(z\overline{a})$$
$$|1-\overline{a}z|^2=1^2+|\overline{a}z|^2-2\mathrm{Re}(a\overline{z})=1+|a|^2-2\mathrm{Re}(\overline{z\overline{a}})$$
$$\Rightarrow\left|\frac{z-a}{1-\overline{a}z}\right|=\sqrt{\frac{|z-a|^2}{|1-\overline{a}z|^2}}=1$$
$$(2)\ |z-a|^2=|z|^2+|a|^2-2\mathrm{Re}(z\overline{a})$$
$$|1-\overline{a}z|^2=1^2+|a|^2|z|^2-2\mathrm{Re}(\overline{z\overline{a}})$$
$$\text{由}\ |z|,|a|<1\text{，知}$$
$$\frac{|z-a|}{|1-\overline{a}z|}=\sqrt{\frac{|z-a|^2}{|1-\overline{a}z|^2}}<\sqrt{\frac{1+|a|^2-2\mathrm{Re}(z\overline{a})}{1+|a|^2-2\mathrm{Re}(z\overline{a})}}=1$$

1.8

$$\text{解}:(1)\text{假设}$$
$$|z_1+\cdots+z_{n-1}|\geq|z_1|-|z_2|-\cdots-|z_{n-1}|\text{成立，则}$$
$$|z_1+z_2+\cdots+z_n|-|z_n|\geq|z_1|-|z_2|-\cdots-|z_{n-1}|-|z_n|$$
$$\text{下证}\ |z_1+z_2+\cdots+z_{n-1}|-|z_n|\leq|z_1+z_2+\cdots+z_n|$$
$$\text{令}\sum_{k=1}^{n-1}z_k=S_{n-1}\text{，则}|S_{n-1}+z_n-z_n|=|S_{n-1}|\leq|S_{n-1}+z_n|+|z_n|$$
$$\text{即}\ |S_{n-1}+z_n|\geq|S_{n-1}|-|z_n|$$
$$\text{即}\ |z_1+z_2+\cdots+z_n|\geq|z_1+z_2+\cdots+z_{n-1}|-|z_n|$$
$$\text{显然}n=2\ \text{时有}\ |z_1+z_2|\geq|z_1|-|z_2|\text{，证毕}$$

1.9

$$\text{解}:(0)\ z_1,z_2,z_3\ \text{共线}$$
$$(0)\Rightarrow(1)\ \text{是显然的}$$
$$(1)\Rightarrow(2):$$
$$\text{令}\frac{z_1-z_2}{z_2-z_3}=t\text{，则}z_1-z_2=tz_2-tz_3$$
$$(x_1-x_2)+i(y_1-y_2)=(tx_2-tx_3)+i(ty_2-ty_3)$$
$$\Rightarrow\begin{cases}x_1-x_2=tx_2-tx_3\\y_1-y_2=ty_2-ty_3\end{cases}$$
$$\text{则}\overline{z_1}z_2+\overline{z_2}z_3+\overline{z_3}z_1=(x_1x_2+y_1y_2+x_2x_3+y_2y_3+x_3x_1+y_3y_1)+i(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3)$$
$$=(x_1x_2+y_1y_2+x_2x_3+y_2y_3+x_3x_1+y_3y_1)+i\left[x_1(y_2-y_3)-(x_2-x_3)y_1+y_2(x_2-x_3)-x_2(y_2-y_3)\right]$$
$$=(x_1x_2+y_1y_2+x_2x_3+y_2y_3+x_3x_1+y_3y_1)\in\mathbb{R}$$
$$(2)\Rightarrow(3):$$
$$\text{已知}(x_1-x_2)(y_2-y_3)=(x_2-x_3)(y_1-y_2)\text{，且有}\lambda_2,\lambda_3\ \text{使得}\lambda_2z_2+\lambda_3z_3=(\lambda_2+\lambda_3)z_1$$
$$\text{即}\begin{cases}\lambda_2x_2+\lambda_3x_3=(\lambda_2+\lambda_3)x_1\\\lambda_2y_2+\lambda_3y_3=(\lambda_2+\lambda_3)y_1\end{cases}\text{，注意到}\begin{cases}\lambda_2=y_1-y_3\\\lambda_3=y_2-y_1\end{cases}\text{即可}$$
$$(3)\Rightarrow(0):$$
$$\frac{z_1-z_2}{z_2-z_3}=\frac{\left(\frac{\lambda_2x_2+\lambda_3x_3}{\lambda_2+\lambda_3}-x_2\right)+i\left(\frac{\lambda_2y_2+\lambda_3y_3}{\lambda_2+\lambda_3}-y_2\right)}{(x_2-x_3)+i(y_2-y_3)}$$
$$=\frac{1}{\lambda_2+\lambda_3}\cdot\frac{(\lambda_3x_3-\lambda_2x_2)+i(\lambda_3y_3-\lambda_2y_2)}{(x_2-x_3)+i(y_2-y_3)}=\frac{-\lambda_3}{\lambda_2+\lambda_3}\in\mathbb{R}$$

注意:这里的公式是用AI识别手写体生成的,由于我使用的方法过于愚蠢又臭又长,我懒得仔细校对,可能有诸多错误,仅参考思路即可.

1.10

$$\text{解：若四点共圆，则}\angle AZ_3B=\angle AZ_4B\ (\text{A为}z_1,\text{B为}z_2)$$
$$\text{Arg}\left(\frac{z_1-z_3}{z_2-z_3}\right)=\angle AZ_3B,$$
$$\text{Arg}\left(\frac{z_1-z_4}{z_2-z_4}\right)=\angle AZ_4B.$$
$$\angle AZ_3B+\angle AZ_4B=\pi\ \text{或}\ 0\Rightarrow\angle AZ_3B-(-\angle AZ_4B)=\pi\ \text{或}\ 0.$$
$$\Rightarrow\text{Arg}(\text{原式})=\pi\ \text{或}\ 0\Rightarrow\text{原式为实数}.$$
$$\text{若原式为实数，则}\arg\left(\frac{z_1-z_3}{z_2-z_3}\Big/\frac{z_1-z_4}{z_2-z_4}\right)=k\pi,\text{则显然四点共圆.}$$

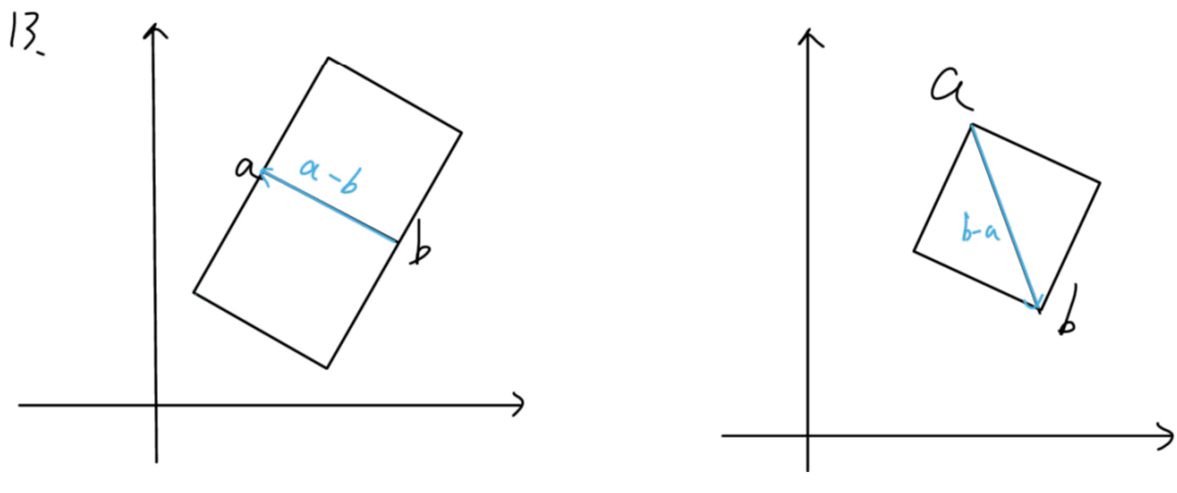
1.11

$$\text{解：}\ z_1+z_2=-z_3\Rightarrow|z_1+z_2|=|-z_3|=1$$
$$\Rightarrow|z_1+z_2|^2=1^2+1^2+2\text{Re}(z_1\overline{z_2})=1\Rightarrow 2\text{Re}(z_1\overline{z_2})=-1$$
$$\Rightarrow|z_1-z_2|^2=1^2+1^2-2\text{Re}(z_1\overline{z_2})=3\Rightarrow|z_1-z_2|=\sqrt{3}$$
$$\text{同理}\ |z_1-z_3|=|z_2-z_3|=\sqrt{3}\text{，故为等边三角形}$$

1.12

$$\text{解：}z_1+z_2=(x_1+x_2)+i(y_1+y_2)$$
$$z_1z_2=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$$
$$\Rightarrow\quad\begin{cases}y_1+y_2=0\\x_1y_2+x_2y_1=0\end{cases}$$
$$①y_1=y_2=0\text{，都为实数}$$
$$②y_1=-y_2\neq0\text{，则}-x_1y_1+x_2y_1=0\Rightarrow x_1=x_2$$
$$\Rightarrow z_1=\overline{z_2}\text{，共轭}$$

1.13



解：本题有三种情况, 请见上图

$$\begin{cases}a+i(b-a)\\b+i(b-a)\end{cases}$$
$$\begin{cases}a+i(a-b)\\b+i(a-b)\end{cases}$$
$$\begin{cases}a+(b-a)\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}i}\\a+(b-a)\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}i}\end{cases}$$

1.14

解：(1)0 (2)0 (3)无

解析略

1.15

解：(1)略

(2)$z_0 \neq 0$及负数时, 结论是对的

$z_0 < 0$时，若z_n从上半面趋近，$\mathrm{Arg} z_n \to -\pi$；反之$\mathrm{Arg} z_n \to \pi$，矛盾。

$z_0 = 0$时，$\arg z_n$不收敛

(3)$z_n \to \infty$，成立，但$\arg z_n$无极限。

1.16

解：(1)点a，b连线的垂直平分线；

(2)焦点为a，b的椭圆；

(3)是与Oy轴相切于原点的圆族及Oy轴（均不包括原点）；

(4)是经过$((\pm 1, 0))$的圆族及(Ox)轴（均不包括点$((1,0)$和$(-1,0))$

(5)是以点$(\pm 1, 0)$为对称点的圆族及Oy轴。

1.17

解：略

1.18

解：$Ax + By = D$，即

$$A\frac{z+\overline{z}}{2} + B\frac{z-\overline{z}}{2i} = D$$

$$A(z+\overline{z}) - Bi(z-\overline{z}) = 2D$$

$$\Rightarrow (A+Bi)\overline{z} + (A-Bi)z = 2D = C$$

1.19

解：(1) $y = x$

(2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(3) $y = \dfrac{1}{x}$

(4) $y = \dfrac{1}{x}$ 的一部分

1.20

解：$x = \dfrac{z+\overline{z}}{2},\ y = \dfrac{z-\overline{z}}{2i}$

$$x^2 + y^2 = z\overline{z},\ 2x = z + \overline{z}$$

$$\Rightarrow z\overline{z} + z + \overline{z} = 1$$

即 $z\overline{z} + z + \overline{z} + 1 = (z+1)(\overline{z}+1) = (z+1)\overline{(z+1)} = 2$

$$\Rightarrow |z+1| = \sqrt{2}$$